Assignment 12

This homework is due Monday April 27.

There are total 55 points in this assignment. 50 points is considered 100%. If you go over 50 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 7.3, 7.4 of Textbook.

(1) [5pt] Find two Laurent series expansions for $f(z) = \frac{1}{z^3 - z^4}$ centered at 0. Where each of them is valid? (*Hint:* For $|z| < 1$, use geometric series for $\frac{1}{1-z}$. For $|z| > 1$, write $\frac{1}{1-z} = \frac{-1}{z}\frac{1}{1-\frac{1}{z}}$ and use geometric series.)

(2) [5pt] Find the Laurent series centered at 0 for the following functions. (*Hint:* Use Taylor series for the involved trig functions.)
   (a) $\frac{\sin 2z}{z^4}$.
   (b) $\frac{\cosh z - \cos z}{z^5}$.
   (c) $\sin \frac{1}{z}$.

(3) [5pt] Find the Laurent series for $f(z) = \frac{1}{z^4(1-z)^2}$ centered at 0 and valid for $|z| > 1$. (*Hint:* Find Laurent series for $\frac{1}{(1-z)^2}$ first. To do that, differentiate the Laurent series for $\frac{1}{1-z}$.)

(4) [5pt] Let a, b be positive real numbers with $b > a > 1$. Find the Laurent series for $\mathrm{Log}\left(\frac{z-a}{z-b}\right)$ centered at 0 and valid for $|z| > b$. (*Hint:* $\mathrm{Log}\left(\frac{z-a}{z-b}\right) = \mathrm{Log}(1 - \frac{a}{z}) - \mathrm{Log}(1 - \frac{b}{z})$. Use Taylor expansion for Log, don't forget to explain why it's valid for the given values of z.)

(5) [10pt] Locate the zeros of the following functions and determine their order.

   (a) $(1 + z^2)^3$.
   (b) $z^6 + 2z^3 + 1$.
   (c) $\sin^2 z$.
   (d) $\sin z^2$.
   (e) $1 + e^z$.
   (f) $z^3 e^{z-1}$.

(6) [10pt] Locate the poles of the following functions and determine their order.

   (a) $(z^2+1)^{-3}(z-1)^{-5}$.
   (b) $(z^4 + z^3 - 2z^2)^{-1}$.
   (c) $z \cot z$.
   (d) $z^{-5} \sin z$.
   (e) $(z^2(1 - \cos z))^{-1}$.
   (f) $(1 - e^z)^{-1}$.

(7) [10pt] Locate the singularities of the following functions and determine their type: removable, pole of order n (find n), essential, non-isolated.

   (a) $\sin \frac{1}{z}$.
   (b) $\tan z$.
   (c) $\frac{z}{\sin z}$.
   (d) $\frac{\sin z}{z^2+z}$.
   (e) $\frac{e^z - 1}{z}$.
   (f) $z^3 e^{\frac{1}{z}}$.

(8) [5pt] Let f have a pole of order k at z_0. Show that f' has a pole of order $k + 1$ at z_0. (*Hint:* Differentiate $f(z) = \frac{h(z)}{(z-z_0)^k}$.)